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The scattering matrix for rapidly oscillating potentials

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Abstract. For Schrödinger Hamiltonians with potentials that are rapidly oscillating near infinity, we show that the T matrix is in the Hilbert–Schmidt class, implying finiteness of the total scattering cross section. We also estimate the high-energy behaviour of the cross section.

In the Hilbert space $L^2(\mathbb{R}^3)$, we consider self-adjoint Schrödinger operators of the form $H = \boldsymbol{P}^2 + V$, where $\boldsymbol{P}$ is the three-component momentum operator and V is the multiplication operator by a real-valued function $v(\boldsymbol{x})$. More precisely, H is assumed to be a self-adjoint extension of the symmetric differential operator $-\Delta + v(\boldsymbol{x})$ defined on some suitable subset of $L^2(\mathbb{R}^3)$. We use the notations of Amrein and Pearson (1979) and set in particular $H_0 = \boldsymbol{P}^2$ and $U_t = \exp(-\mathrm{i}H_0 t)$. If $\hat{f}$ denotes the Fourier transform of f, then correspondence $f \mapsto \{f_\lambda(\boldsymbol{\omega})\}$, where

$$f_\lambda(\boldsymbol{\omega}) = \frac{1}{\sqrt{2}}\lambda^{1/4}\hat{f}(\sqrt{\lambda}\boldsymbol{\omega}) \tag{1}$$

[$\lambda \in (0, \infty)$, $\boldsymbol{\omega} \in S^{(2)} \equiv$ the unit sphere in $\mathbb{R}^3$], defines a unitary map from $L^2(\mathbb{R}^3)$ onto $L^2([0, \infty); L^2(S^{(2)}))$ which diagonalises H_0. Let $S(\lambda)$ denote the scattering matrix at energy λ (i.e. $S(\lambda)$ is an operator in $\mathscr{H}_0 \equiv L^2(S^{(2)})$). The total scattering cross section, averaged over all initial directions, for scattering of a beam of non-relativistic particles at energy λ off the potential $v(\boldsymbol{x})$ is given by

$$\bar{\sigma}(\lambda) = \pi\lambda^{-1}\|R(\lambda)\|_{\mathrm{HS}}^2 \tag{2}$$

where $R(\lambda) = S(\lambda) - I_0$, I_0 is the identity operator in $\mathscr{H}_0$ and $\|A\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm of the operator A (see Amrein and Pearson (1979) for details).

In a previous paper (Amrein and Pearson 1979) we introduced a time-dependent method of estimating the averaged total scattering cross section $\bar{\sigma}(\lambda)$. We found that $\bar{\sigma}(\lambda)$ is finite provided the potential $v(\boldsymbol{x})$ tends to zero more rapidly than $|\boldsymbol{x}|^{-2}$ as $|\boldsymbol{x}| \to \infty$, whereas for more slowly decreasing potentials $\bar{\sigma}(\lambda)$ is, in general, infinite at all energies λ. In the present paper we apply this method to potentials that are rapidly oscillating near infinity. The amplitude of oscillation may decrease to zero very slowly or even diverge. We obtain the existence of the wave operators, the finiteness at almost all energies of $\|R(\lambda)\|_{\mathrm{HS}}$, and hence of $\bar{\sigma}(\lambda)$, as well as a bound on the high-energy behaviour of $\bar{\sigma}(\lambda)$. In proposition 1 we consider a simple class of spherically symmetric

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potentials $v(|\boldsymbol{x}|)$, and then discuss some possible generalisations. For other results on oscillating potentials we refer to Chadan and Martin (1979), Pearson (1979) and the papers quoted there.

In what follows, ρ will always be a continuously differentiable function from $(0, \infty)$ to $\mathbb{R}$ vanishing near 0 and near ∞. For each such ρ, we denote by $P(\rho)$ the orthogonal projection onto the following subspace $\mathscr{H}(\rho)$ of $L^2(\mathbb{R}^3)$:

$$\mathscr{H}(\rho)=\{f|f_\lambda=\rho(\lambda)g \text{ with } g\in\mathscr{H}_0\}. \tag{3}$$

We shall use the following three results which are immediate consequences of those contained in § 3 of Amrein and Pearson (1979).

(i) Let Φ be the multiplication operator in $L^2(\mathbb{R}^3)$ by a function $\phi(\boldsymbol{x})$ which is twice differentiable and satisfies $\phi(\boldsymbol{x})\to 1$ as $|\boldsymbol{x}|\to\infty$. Then

$$\int_{-\infty}^{\infty} \mathrm{d}\lambda\,\rho(\lambda)^2\|R(\lambda)\|_{\mathrm{HS}}^2 \leqslant \left(\int_{-\infty}^{\infty} \mathrm{d}t\,\|(H\Phi-\Phi H_0)U_tP(\rho)\|_{\mathrm{HS}}\right)^2. \tag{4}$$

(ii) The finiteness of the integral on the RHS of (4), for each ρ in the class indicated above, implies the existence of the wave operators

$$\Omega_\pm(H,H_0)=s-\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}Ht}\,\mathrm{e}^{-\mathrm{i}H_0t}. \tag{5}$$

(iii) Let W be the multiplication operator by a real-valued function $w(\boldsymbol{x})$ satisfying, for some $0\leqslant M<\infty$ and some $\nu>\frac{1}{2}$,

$$\int_{|\boldsymbol{x}|\geqslant M} \mathrm{d}^3x\,(1+|\boldsymbol{x}|)^{2\nu}|w(\boldsymbol{x})|^2<\infty. \tag{6}$$

Then

$$\int_{-\infty}^{\infty} \mathrm{d}t\,\|WU_tP(\rho)\|_{\mathrm{HS}}+\sum_{k=1}^{3}\int_{-\infty}^{\infty} \mathrm{d}t\,\|WP_kU_tP(\rho)\|_{\mathrm{HS}}<\infty. \tag{7}$$

Proposition 1. Let $v:(0,\infty)\to\mathbb{R}$ be such that, for some finite r_0 and all $r\geqslant r_0$:

$$|v(r)|\leqslant cr^\alpha \qquad \text{and} \qquad \left|\int_r^\infty \mathrm{d}s\,v(s)\right|\leqslant cr^{-\beta} \tag{8}$$

with $\beta>2$ and $\beta-\alpha>3$. Then (a) the wave operators $\Omega_\pm(H,H_0)$ exist; (b) for each admissible ρ:

$$\int_0^\infty \mathrm{d}\lambda\,\rho(\lambda)^2\|R(\lambda)\|_{\mathrm{HS}}^2<\infty; \tag{9}$$

(c) for each $\epsilon>0$,

$$\int_1^\infty \mathrm{d}\lambda\,\lambda^{-1-\epsilon}\bar{\sigma}(\lambda)<\infty. \tag{10}$$

Proof. Let $\psi:[0,\infty)\to\mathbb{R}$ be a C^∞ function such that $\psi(r)=0$ for $0\leqslant r\leqslant r_0$ and $\psi(r)=1$ for $r\geqslant 2r_0$. Define Φ to be the multiplication operator by the function $\phi(|\boldsymbol{x}|)$, where

$$\phi(r)=\psi(r)\left[1+\int_r^\infty \mathrm{d}u\int_u^\infty \mathrm{d}s\,v(s)\right]. \tag{11}$$

Now

$$\phi'(r)=\psi'(r)\left[1+\int_r^\infty \mathrm{d}u\int_u^\infty \mathrm{d}s\, v(s)\right]-\psi(r)\int_r^\infty \mathrm{d}s\, v(s).$$

By (8), and since ψ' has compact support in $(0, \infty)$, we have

$$|\phi'(r)|\leq c_1(1+r)^{-\beta}. \tag{12}$$

Furthermore:

$$\begin{aligned} w_0(r)&\equiv-\phi''(r)-\frac{2}{r}\phi'(r)+v(r)\phi(r)\\ &=-\psi''(r)\left[1+\int_r^\infty \mathrm{d}u\int_u^\infty \mathrm{d}s\, v(s)\right]+2\psi'(r)\int_r^\infty \mathrm{d}s\, v(s)\\ &\quad-\frac{2}{r}\phi'(r)+v(r)\psi(r)\int_r^\infty \mathrm{d}u\int_u^\infty \mathrm{d}s\, v(s). \end{aligned} \tag{13}$$

By (8) and (12), the absolute value of the last summand in (13) is majorised by

$$\begin{aligned} |\psi(r)|cr^{\alpha}\left|\int_r^\infty \mathrm{d}u\, cu^{-\beta}\right| &\leq c_2|\psi(r)|r^{-(\beta-\alpha-1)}\\ &\leq c_3(1+r)^{-(\beta-\alpha-1)}. \end{aligned}$$

Hence

$$|w_0(r)|\leq c_4(1+r)^{-\gamma} \tag{14}$$

with $\gamma=\min(\beta, \beta-\alpha-1)>2$.

Now

$$\begin{aligned} &\|(H\Phi-\Phi H_0)U_tP(\rho)\|_{\mathrm{HS}}\\ &\qquad=\|[-\Delta\Phi+V\Phi-2\mathrm{i}(\nabla\Phi)\,.\,P]U_tP(\rho)\|_{\mathrm{HS}}\\ &\qquad\leq\|W_0U_tP(\rho)\|_{\mathrm{HS}}+2\sum_{k=1}^{3}\|W_kP_kU_tP(\rho)\|_{\mathrm{HS}} \end{aligned} \tag{15}$$

where W_0 is the multiplication operator by $w_0(|\boldsymbol{x}|)$ and W_k that by $w_k(\boldsymbol{x})\equiv x_k|\boldsymbol{x}|^{-1}\phi'(|\boldsymbol{x}|)$. Since w_0 and w_k all satisfy the condition (6), we obtain from (15) and (7) that

$$\int_{-\infty}^{\infty}\mathrm{d}t\|(H\Phi-\Phi H_0)U_tP(\rho)\|_{\mathrm{HS}}<\infty.$$

In view of (4), this proves (*b*). The existence of $\Omega_\pm(H, H_0)$ follows from (ii), and the high-energy bound (10) is obtained as in theorem 3 of Amrein and Pearson (1979) (the dominating term at large λ is again that arising from $\Sigma_k\,\|W_kP_kU_tP(\rho)\|_{\mathrm{HS}}$.

The idea of the proof of proposition 1 is to construct the multiplication operator Φ from a function ϕ which may be regarded, for large r, as an approximate solution of the zero-energy Schrödinger equation:

$$-\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2}+v\phi=0. \tag{16}$$

Thus, for $r > 2r_0$, equation (11) implies $\mathrm{d}^2\phi/\mathrm{d}r^2 = v$ which, with $\phi \to 1$ as $r \to \infty$, gives an approximate solution to (16). Proposition 1 may be extended to a wider class of potentials by obtaining more detailed estimates of the solutions of (16).

Let

$$w(r) = -\int_r^\infty \mathrm{d}s\, v(s) \tag{17}$$

and consider the pair of coupled integral equations

$$\begin{aligned}\phi_0(r) &= 1 + r\int_r^\infty \mathrm{d}s\, w(s)\theta(s) - \int_r^\infty \mathrm{d}s\, w(s)[\phi_0(s) + s\theta(s)] \\ \theta(r) &= w(r)\phi_0(r) + \int_r^\infty \mathrm{d}s\, w(s)\theta(s).\end{aligned} \tag{18}$$

Differentiating the first equation gives

$$\mathrm{d}\phi_0/\mathrm{d}r = \theta \tag{19}$$

and differentiating the second, using (17) and (19), gives

$$\mathrm{d}\theta/\mathrm{d}r = v\phi_0 \tag{20}$$

so that ϕ_0 is a solution of (16). Moreover $\phi_0 \to 1$ as $r \to \infty$.

To solve (18) for sufficiently large r, we shall suppose that $w(|\boldsymbol{x}|)$ satisfies (6) in which case, for some $\epsilon > 0$,

$$\int_M^\infty \mathrm{d}s\, s^{3+\epsilon}|w(s)|^2 < \infty. \tag{21}$$

Estimates of solutions of (18) may more readily be carried out by means of the substitution

$$\chi(r) = \theta(r) - w(r)\phi_0(r) \tag{22}$$

in which case we have

$$\begin{aligned}\phi_0(r) = 1 &+ r\int_r^\infty \mathrm{d}s\, w(s)\chi(s) - \int_r^\infty \mathrm{d}s\, w(s)[\phi_0(s) + s\chi(s)] \\ &+ r\int_r^\infty \mathrm{d}s\, w^2(s)\phi_0(s) - \int_r^\infty \mathrm{d}s\, w^2(s)s\phi_0(s) \\ \chi(r) &= \int_r^\infty \mathrm{d}s\, w(s)\chi(s) + \int_r^\infty \mathrm{d}s\, w^2(s)\phi_0(s).\end{aligned} \tag{23}$$

Equation (23) may now be iterated by taking

$$\phi_0^{(0)} = 1 \qquad \chi^{(0)} = 0$$

$$\begin{aligned}\phi_0^{(n+1)} = 1 &+ r\int_r^\infty \mathrm{d}s\, w(s)\chi^{(n)}(s) - \int_r^\infty \mathrm{d}s\, w(s)[\phi_0^{(n)}(s) + s\chi^{(n)}(s)] \\ &+ r\int_r^\infty \mathrm{d}s\, w^2(s)\phi_0^{(n)}(s) - \int_r^\infty \mathrm{d}s\, w^2(s)s\phi_0^{(n)}(s) \\ \chi^{(n+1)}(r) &= \int_r^\infty \mathrm{d}s\, w(s)\chi^{(n)}(s) + \int_r^\infty \mathrm{d}s\, w^2(s)\phi_0^{(n)}(s).\end{aligned}$$

On using Schwarz's inequality with (21), we have

$$\int_r^\infty ds|w(s)| = \int_r^\infty ds[s^{(3+\epsilon)/2}w(s)]s^{-(3+\epsilon)/2} \leqslant \text{constant} \times r^{-1-\epsilon/2}.$$

Also

$$\int_r^\infty ds\, w^2(s) \leqslant \frac{1}{r^{3+\epsilon}} \int_r^\infty ds\, s^{3+\epsilon} w^2(s).$$

Hence

$$\phi_0^{(1)}(r) = 1 + O\left(\frac{1}{r^{1+\epsilon/2}}\right) \qquad \text{as} \qquad r \to \infty$$

whereas

$$\chi^{(1)}(r) = O\left(\frac{1}{r^{3+\epsilon}}\right).$$

One may now deduce inductively from (23) that, for given R, constants A_n and B_n (depending on R) can be found such that for $r > R$

$$|\phi_0^{(n+1)}(r) - \phi_0^{(n)}(r)| \leqslant A_n r^{-1-\epsilon/2}$$
$$|\chi^{(n+1)}(r) - \chi^{(n)}(r)| \leqslant B_n r^{-3-\epsilon}$$

and

$$(A_{n+1} + B_{n+1}) \leqslant \text{constant} \times r^{-1-\epsilon/2}(A_n + B_n).$$

Thus, if R is chosen so large that constant $\times R^{-1-\epsilon/2} < 1$, we see that the series $\Sigma_{n=0}^\infty A_n$ and $\Sigma_{n=0}^\infty B_n$ are convergent, and we may deduce that the iteration converges to a pair of functions ϕ_0, χ for which, as $r \to \infty$,

$$\phi_0(r) = 1 + O\left(\frac{1}{r^{1+\epsilon/2}}\right)$$
$$\chi(r) = O\left(\frac{1}{r^{3+\epsilon}}\right). \tag{24}$$

From (19) and (22) we obtain the estimates

$$\phi_0(r) = 1 + O\left(\frac{1}{r^{1+\epsilon/2}}\right)$$
$$\frac{d\phi_0(r)}{dr} = w(r)\left[1 + O\left(\frac{1}{r^{1+\epsilon/2}}\right)\right] + O\left(\frac{1}{r^{3+\epsilon}}\right). \tag{25}$$

We may now carry through the argument of the proof of proposition 1, replacing (11) by $\phi(r) = \psi(r)\phi_0(r)$ and taking $r_0 \geqslant \max(M, R)$. We then have:

Proposition 2. Let $v : (0, \infty) \to \mathbb{R}$ be such that (21) holds for some finite M, where $w(r)$ is defined by (17) (as an improper Riemann integral). Then the conclusions (a) and (b) of proposition 1 follow.

Remark 1. To estimate high-energy behaviour of cross sections, we have to replace, as in our previous paper, the condition (21) by $|w(r)| \leq \text{constant} \times r^{-2-\epsilon} (\epsilon > 0)$ for $r > R$. Under this condition, conclusion (c) of proposition 1 also follows.

Examples. Let $v(r) = cr^{\kappa} \sin(r^{\gamma})$. Proposition 1 applies provided that $\gamma - \kappa > 3$ and $\gamma - 2\kappa > 4$, and proposition 2 applies if $\gamma - \kappa > 3$. Choosing some particular values for κ, we obtain the following examples in which γ must be larger than γ_1 if one wants to apply proposition 1 and larger than γ_2 if one uses proposition 2:

$$v(r) = \frac{c}{r} \sin(r^{\gamma}) \qquad \text{with } \gamma_1 = \gamma_2 = 2$$

$$v(r) = c \sin(r^{\gamma}) \qquad \text{with } \gamma_1 = 4, \gamma_2 = 3$$

$$v(r) = 1066 r^{1984} \sin(r^{\gamma}) \qquad \text{with } \gamma_1 = 3972, \gamma_2 = 1987.$$

Remark 2. The above methods may be modified to apply to a class of non-spherical potentials having the form, for $|\boldsymbol{x}| > R$,

$$v(\boldsymbol{x}) = v_0(\boldsymbol{x}) \operatorname{div} \boldsymbol{h}(\boldsymbol{x}) \tag{26}$$

where v_0 is a scalar potential and $\boldsymbol{h}$ is a vector potential satisfying curl $\boldsymbol{h} = 0$. (This condition is always satisfied when $\boldsymbol{h}(\boldsymbol{x}) = g(|\boldsymbol{x}|)\boldsymbol{x}$.)

With ψ defined as before, replace (11) by

$$\phi(\boldsymbol{x}) = \psi(\boldsymbol{x})[1 + \mu(\boldsymbol{x}) v_0(\boldsymbol{x})] \tag{27}$$

where grad $\mu = \boldsymbol{h}$. Then

$$-\Delta\phi + v\phi = \psi(-v_0\Delta\mu - \mu\Delta v_0 - 2\boldsymbol{h} \cdot \operatorname{grad} v_0 + v + v\mu v_0) + Z$$

$$= \psi(-\mu\Delta v_0 - 2\boldsymbol{h} \cdot \operatorname{grad} v_0 + v\mu v_0) + Z$$

where we have used the result that $\Delta\mu = \operatorname{div} \boldsymbol{h}$ together with (26), and where Z is a sum of additional terms having compact support.

Now assume that $\mu v_0 \to 0$ as $|\boldsymbol{x}| \to \infty$ and that

$$F \equiv |\mu\Delta v_0| + |\boldsymbol{h} \cdot \operatorname{grad} v_0| + |v\mu v_0| + |\mu \operatorname{grad} v_0| + |v_0 \boldsymbol{h}|$$

satisfies $F(\boldsymbol{x}) \leq c|\boldsymbol{x}|^{-\beta}$ for $|\boldsymbol{x}| \geq r_0$ and some $\beta > 2$. Then the conclusions of proposition 1 follow.

As an example of a non-spherical potential for which we may deduce finiteness of total cross section in this way we have, with $\epsilon > 0$,

$$v(\boldsymbol{x}) = v_0(\boldsymbol{x}) \cos(r^{3+\epsilon})$$

provided, for example, v_0 and its derivatives up to second order are bounded.

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